THE SIMPLEX METHOD FOR QUADRATIC PROGRAMMING

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SUMMARY

A computational procedure is given for finding the minimum of a quadratic function of variables subject to linear inequality constraints. The procedure is analogous to the Simplex Method for linear programming, being based on the Barankin-Dorfman procedure for this problem.

CONTENTS

THE SIMPLEX METHOD FOR QUADRATIC PROGRAMMING

1. INTRODUCTION

In this paper, by "quadratic programming" we shall understand the problem of determining values of several real variables, subject to linear inequality constraints, which yield the extreme value of a quadratic function. Besides being a step on the way toward solution of the elaborate non-linear programming problems which economic models often present, a usable computational procedure for quadratic programming can be applied to a number of problems of interest in themselves:

Regression. To find the best least-squares fit to given data, where certain parameters are known a priori to satisfy inequality constraints (e.g., being non-negative).

Efficient production. Maximization of profit, assuming linear production functions and linearly varying marginal costs.

"Portfolio" problem.* To find a combination of random variables having given expectation and minimum variance.

Convex programming. To find the minimum of a general convex function under linear constraints using a quadratic approximation.

Let the variables of the problem constitute the n-vector $x = (x_1, .., x_n)'$ (' will denote transposition; we take x to be a column vector, that is, an n-by-1 matrix). Letting A be

*Due to Markowitz [6].

an m by n matrix, and b m-by-1, we will express the linear constraints of the problem by

(1) $x \geq 0$, $Ax = b$,

that is,

$$x_j \geq 0 \ (j=1, \ldots, n), \ \sum_{j=1}^{n} a_{ij}x_j = b_i \ (i=1, \ldots, m).$$

We recall that any combination of linear equation and inequality constraints can be written in the form (1) if additional variables are allowed into the problem.

Let p be a 1-by-n matrix, and C be an n-by-n symmetric matrix. We shall write the <u>objective function</u>, the (partly) quadratic function to be extremized subject to (1), as

(2) $f(\lambda, x) = \lambda px + 1/2\, x'Cx$,

or

$$f(\lambda, x) = \lambda \sum_j p_j x_j + 1/2 \sum_{j,k} x_j C_{jk} x_k,$$

where λ, a single non-negative real parameter, can be chosen as convenient. The problem can now be stated as:

The quadratic problem for $\lambda \geq 0$:

(3) Minimize $f(\lambda, x) = \lambda px + 1/2\, x'Cx$

subject to $x \geq 0$, $Ax = b$.

An important restriction must be placed on the quadratic part, C, of the objective function in order to ensure the success of the computational method: the function f must be convex, that is, C must be positive semidefinite. This

condition - apparently essential, in the present state of the art, to all non-linear programming schemes - ensures that any <u>local</u> minimum encountered in the problem will be the <u>global</u> solution desired. Algebraically, the assertion of positive semidefiniteness for C is that

(4) $x'Cx \geq 0$ for all x.

In economic problems, it is the ascription of non-increasing returns to scale for all activities, since the marginal cost of changing from the program x to the program $x + \Delta x$ is given by

$$\frac{d}{dt} f(\lambda, x+t\Delta x) = p\Delta x + xC\Delta x + t\Delta x'C\Delta x,$$

which will be a non-decreasing function of t. We shall assume it from now on. A more detailed discussion of the role of this property in quadratic programming is given in [4].

A number of proposals for the computational solution of quadratic programming problems have been made in the last two years; those which seem suited to high-speed digital computers are given in References [1] - [6] below. Barankin and Dorfman [1] first pointed out the linear formulation of the quadratic problem, inspiring the present approach; our Section 2 is taken, with changed notation, from Sections 1 and 3 of [1].

The principal respect in which the present method differs from these is in using only the computational mechanisms of the Simplex Method for linear programming problems. It is therefore a simple matter to convert a computing machine code for

linear programming into one for quadratic programming; the SHARE linear programming code for the IBM 704 requires modification in eleven instructions for this purpose.

In the sequel this method is developed in two forms, "short" and "long." The computations of the long form are like those of the short, but are aimed at solving the quadratic problem (3) for <u>all</u> $\lambda \geq 0$ and at the same time avoiding certain restrictions on the use of the short form. The table below summarizes the conditions for use of these methods. The estimate of the number of Simplex Method changes-of-basis needed to solve the problem is based on experience like that described in Section 6.

Solution of (3) by -

	Short Form	**Long Form**
Conditions	Either $\lambda = 0$ <u>or</u> C positive definite	C positive semidefinite
Solution obtained for	fixed λ	all $\lambda \geq 0$
Size of equivalent linear program	up to m+n equations, m+3n variables	up to m+n equations, m+3n+1 variables
Estimated number of basis changes for solution	2(m+n)	4(m+n)

2. PRELIMINARIES

Since we are interested, in part, in the solution of the quadratic problem for all $\lambda \geq 0$, let us define

$$\text{for } \lambda \geq 0: F(\lambda) = \text{Min}\left\{\lambda px + 1/2\, x'Cx \,;\, x \geq 0,\ Ax = b\right\}.$$

Quite a bit of information about $F(\lambda)$ can be obtained without calculating it. Throughout it will be assumed, of course, that there exist <u>feasible</u> x, i.e. $x \geq 0$ such that $Ax = b$. Nevertheless, we may have $F(\lambda) = -\infty$ for some (and hence all) $\lambda > 0$.

First, an important feature of the positive semi-definiteness of C is given by

<u>LEMMA 1</u>: $x'Cx = 0$ implies $Cx = 0$.

Proof: For any n-vector y we have $0 \leq (y+tx)'C(y+tx) = y'Cy + 2ty'Cx$ for any number t, whence $y'Cx = 0$; $Cx = 0$ follows at once.

This leads to

<u>LEMMA 2</u>: For any $\lambda \geq 0$, the "solution set" of all feasible x such that $f(\lambda,x) = F(\lambda)$ is the intersection of a linear manifold with the constraint set, and px is constant on this set for $\lambda > 0$.

Proof: Let feasible points x, y be given such that $f(\lambda,x) = f(\lambda, y) = F(\lambda)$. Letting $w = y-x$, for any $0 \leq t \leq 1$ the point $x+tw$ belongs to the constraint set; since f is

convex, and $f(\lambda, x+tw)$ is minimal for $t = 0$ and $t = 1$, we have $f(\lambda, x+tw) = f(\lambda, x)$ for all $0 \le t \le 1$, or $\lambda p(x+tw) + 1/2\,(x+tw)'C(x+tw) = \lambda px + 1/2\,x'Cx$, which simplifies to

$$(\lambda pw + x'Cw)t + 1/2\,w'Cw\,t^2 = 0 \quad [0 \le t \le 1].$$

Thus $w'Cw = 0$, whence by lemma 1

$$(5) \quad \begin{cases} Cw = 0, \text{ and hence} \\ pw = 0. \end{cases}$$

Conversely, it is clear that if $f(\lambda, x) = F(\lambda)$, w satisfies (5), and $x + tw$ is feasible, then $f(\lambda, x+tw) = F(\lambda)$; so that the complete solution set for λ given is the intersection of the constraint set with the linear manifold $\{x + tw\}$. Equation (5) yields finally $px = py$ for any two solutions.

If now for any $\lambda > 0$ we choose a feasible solution x_λ such that $f(\lambda, x_\lambda) = F(\lambda)$, by lemma 2 the value px_λ is independent of the choice of x_λ.

<u>Theorem 1</u>: For $\lambda \ge 0$, $F(\lambda)$ is a concave function; px_λ is monotone non-increasing; and x_λ is a solution y of the problem:

$$\text{Min}\left\{ y'Cy \;:\; y \ge 0,\ Ay = b,\ py \le px_\lambda \right\}.$$

Proof: Since $f(\lambda, x)$ is linear in λ, the function $F(\lambda)$ is the infimum of a family of linear functions, and hence concave.

The trend of px_λ is an instance of a quite general fact. Take any λ and μ. Since x_λ minimizes $f(\lambda,x)$, we have

$$\lambda px_\lambda + 1/2\, x_\lambda' Cx_\lambda \leq \lambda px_\mu + 1/2\, x_\mu' Cx_\mu ;$$

and since x_μ minimizes $f(\mu,x)$, we have

$$\mu px_\mu + 1/2\, x_\mu' Cx_\mu \leq \mu px_\lambda + 1/2\, x_\lambda' Cx_\lambda .$$

Adding these inequalities and rearranging, we get

$$(\mu-\lambda)px_\mu \leq (\mu-\lambda)px_\lambda ,$$

which yields $px_\mu \leq px_\lambda$ for $\mu > \lambda$.

Finally, since x_λ does minimize $\lambda px + 1/2\, x'Cx$, any y such that $y'Cy<x_\lambda'Cx_\lambda$ will give $py>px_\lambda$, which proves the last statement.

The next theorem characterizes x_λ in such a way that we will be able to compute it. Only the sufficiency of this condition for the minimization of $f(\lambda,x)$ is needed, since its necessity will follow when we have established that results of the computational scheme of the next section meet this condition if the minimum exists.

<u>THEOREM 2</u>: If $x\geq 0$, $Ax = b$, and there exist $v\geq 0$ (v is n by 1) and u (u is m by 1) such that

(6) $v'x = 0$

and

(7) $Cx - v + A'u + \lambda p' = 0,$

then x solves the problem $\mathrm{Min}\left\{\lambda px + 1/2\ x'Cx : x \geq 0,\ Ax = b\right\}$.

Proof: Let any $y \geq 0$, $Ay = b$ be given. We shall show that $f(\lambda,y) \geq f(\lambda,x)$. From the positive semi-definiteness of C we have

$$(y-x)'\ C(y-x) \geq 0,$$

whence $$y'Cy + x'Cx \geq 2x'Cy$$

or $$y'Cy - x'Cx \geq 2x'C(y-x)$$

so that

$$f(\lambda,y) - f(\lambda,x) = \lambda p(y-x) + 1/2y'Cy - 1/2\ x'Cx$$

$$\geq (\lambda p+x'C)(y-x).$$

Since by (7)

$$\lambda p+x'C = v'-u'A,$$

$$f(\lambda,y) - f(\lambda,x) \geq v'y - v'x - u'Ay + u'Ax$$

$$= v'y - 0 - u'b + u'b \quad \text{(by (6) and feasibility)}$$

$$= v'y \geq 0 \qquad \text{(since } v,y \geq 0\text{)}.$$

The conditions (6) and (7) – especially as necessary, rather than sufficient, conditions – are essentially those of the "saddlepoint" theorem of Kuhn and Tucker [9]. In the present form, the result is due to Barankin and Dorfman [1, Section 3]. The theorem in fact obtains if $f(\lambda,x)$ is replaced by any convex, differentiable function and $Cx + \lambda p'$ of (7) is replaced by its gradient.

The remarkable feature of the quadratic problem is the linearity of the gradient of $f(\lambda,x)$, which confines the non-linearity of the Kuhn-Tucker conditions to the relation (6), $v'x = 0$, which has this sort of combinatorial expression:

(8) $v_j > 0$ implies $x_j = 0$ $(j=1,..,n)$.

In order to explore the relation of the constraints (8) to the linear relations (7), and see how they may be handled numerically, we must consider briefly the main features of the Simplex Method for linear programming [7].

It is required to minimize the linear form cx under the constraints $Dx=e$, $x\geq 0$ (D is p by q, c is 1 by q, e is p by 1). We suppose this problem <u>feasible</u>, i.e., the existence of x satisfying the constraints. It is easy to show through linear dependence that there exists a feasible x having no more than p positive components. A collection of p columns from D which correspond to the non-vanishing components of a feasible x is called a <u>basis</u>, and x is called a <u>basic solution</u>. In the Simplex Method one works always with such bases; given any, it is shown that either (i) the associated basic solution yields the minimum value of the linear form, or (ii) another basis differing in only one column from the given basis can be found whose associated basic solution yields a smaller value for the form, or (iii) one column can be adjoined to the basis such that a sequence of feasible x's associated with these $p+1$ columns can be found on which $cx \to -\infty$.

Thus a sequence of bases is generated which terminates in either a finite or infinite solution of the problem. It is convenient to make an assumption of "non-degeneracy" regarding the constraints of the problem: That any feasible vector x has at least p positive components. A consequence of this assumption is the linear independence of the columns of any basis. It has been shown [7] that every set of constraints can be dealt with so as to be non-degenerate. In the sequel we rely on these results, assuming non-degeneracy in the few places it is necessary.

Returning to the quadratic problem, the conditions that the n-vector x solve the quadratic problem for $\lambda \geq 0$ may be written together as (omitting for the moment $v'x = 0$)

$$\begin{aligned} &Ax && = b, \\ &Cx - v + A'u + p'\lambda && = 0, \\ &x \geq 0,\ v \geq 0, && \end{aligned}$$

or in detached coefficient form as

$$(9)\qquad \begin{array}{cccc|c} x\geq 0 & v\geq 0 & u & \lambda & \\ \hline A & 0 & 0 & 0 & = b \\ C & -I & A' & p' & = 0, \end{array}$$

constituting m + n equations in 2n non-negative variables and m unrestricted variables (λ is not considered a variable). We will be concerned below with the basic solutions of this system. Note that the m columns associated with the

unrestricted variable u are taken to be in every basis (this technical device makes it unnecessary algebraically to eliminate the u's to bring the system to standard form), leaving only n positive variables in any basic solution to go to x and v.

Assuming for the moment the converse of Theorem 2, if the quadratic problem has a solution, then there exist x,v,u satisfying (6) and (7). But (6) implies that at least n components from the 2n x and v vanish; and this establishes the important result of Barankin and Dorfman [1] that some basic solution of (9) constitutes a solution of the quadratic problem.

Since the computational step of the Simplex Method can be used to explore basic solutions, Barankin and Dorfman have accordingly proposed use of the method, beginning with an arbitrary basic solution of (9), to reduce v'x to zero. One method which accomplishes this is given in [4], but it is more complicated, and probably slower, than the present algorithm.

Markowitz, on the other hand, has suggested a method [6] for the "portfolio" problem (equivalent to solving the quadratic problem for all $\lambda \geq 0$) which begins with constraints looser than (9) and which, while retaining (8) v'x = 0, should alter the variables until (9) obtains. The method described here exploits this ingenious idea, differing from the proposal of [6] in keeping to a linear programming format.

3. THE COMPUTATION

Here we present the computational algorithms for the minimization of $\lambda px + 1/2\ x'\ Cx$ subject to $x \geq 0$, $Ax = b$. First is given the "short form," for λ fixed, whose convergence requires that either $\lambda = 0$ or that C be positive definite; next is given the "long form," solving the quadratic problem "parametrically" for all $\lambda \geq 0$, which does not need C positive definite, but which involves two recursions of the "short form" type.

We will suppose below, relying on [7], that the constraints $Ax = b$ and the constraints employed below are all non-degenerate.

SHORT FORM

Let z^1, z^2, and w be respectively n-, n-, and m—component vectors. We begin with the set of relations

$$(10) \qquad Ax \qquad\qquad\qquad + w = b$$

$$(11) \qquad Cx - v + A'u + z^1 - z^2 = -\lambda p'$$

$$(12) \qquad x,\ v,\ z^1,\ z^2,\ w \geq 0,$$

a weakening of the set (9).

Initiation

Since $b \geq 0$, an initial basis for this system can be formed from the coefficients of z^1, z^2, and w. Use the simplex method to minimize

(13) $\qquad \sum_i w_i$

to zero, keeping v and u zero. Discard w and the unused components of z^1, z^2; let the remaining n components be denoted by Z, and their coefficients by E. We now have a solution of the system

$$(14) \qquad \begin{aligned} Ax \qquad\qquad\qquad\qquad &= b \\ Cx - v + A'u + EZ &= -\lambda p' \\ x, v, Z \geq 0. \end{aligned}$$

Recursion

Given a basis and basic solution satisfying (14), (8) $v'x = 0$, and $\sum_{k=1}^{n} Z_j > 0$, make one change of basis in the Simplex procedure for minimizing the linear form

$$(15) \qquad \sum_{k=1}^{n} Z_k,$$

under the side condition

(16) for $k = 1,..,n$: if x_k is in the basis, do not admit v_k; and if v_k is in the basis, do not admit x_k.

Termination

If the form (15) is positive, repeat the recursive step. The form will vanish in at most $\binom{3n}{n}$ iterations, yielding $Z = 0$. The x-part of the terminal basic solution is a solution of the quadratic programming problem for λ.

LONG FORM

Initiation

Having performed the short form computation for $\lambda = 0$, add p' to the short form data, obtaining the system

(17) $Ax \qquad\qquad\qquad\qquad = b$

(18) $Cx - v + A'u + \mu p' + EZ = 0$

and an initial solution having $\mu = 0$, $Z = 0$, and $v'x = 0$.

Recursion

Given a basis and basic solution satisfying (17), (18), $v'x = 0$, and having $Z = 0$, make one change of basis (if possible) in the Simplex procedure for minimizing the linear form

(19) $-\mu$

under the side condition (16) and

(20) do not allow any Z_j in the basis.

Termination

If it is not possible to make the basis change of the recursion, then $\mu = 0$, $F(\lambda) = -\infty$ for all $\lambda > 0$, and a set of feasible x can be found (in Section 5) on which $f(\lambda, x) \rightarrow -\infty$ for $\lambda > 0$.

Otherwise the recursion will yield the finite sequence of values $0 = \mu^0 < \mu^1 < .. < \mu^K$ and the x-parts $x^0, x^1, .., x^K$

of their associated basic solutions, terminating in at most $\binom{2n}{n}$ iterations with the vector x^{∞}, such that:

for $\mu^k \leq \lambda \leq \mu^{k+1}$ the quadratic problem for λ is solved by

$$(21) \quad x = \frac{\mu^{k+1} - \lambda}{\mu^{k+1} - \mu^k} x^k + \frac{\lambda - \mu^k}{\mu^{k+1} - \mu^k} x^{k+1},$$

and for $\lambda \geq \mu^K$ it is solved by

$$(22) \quad x = x^K + (\lambda - \mu^K) x^{\infty} .$$

<u>NOTE:</u> E. M. L. Beale recently communicated an elegant modification of the "short form" procedure above which permits its use in the case that the quadratic form is only positive semidefinite instead of definite. It consists essentially in calculating the effect of a "virtual perturbation" of C which involves replacing C_{jj} by $C_{jj}+\delta^j$, $j = 1,..,n$, for arbitrarily small δ, so that the algorithm can operate as if a positive definite form were employed.

4. EXAMPLE

As an example for calculating with both the short form and the long form, we shall solve the problem

$$\text{Min } 1/2(x_1^2+x_2^2+x_3^2) + \lambda(x_1-2x_3)$$

subject to $x_1, x_2, x_3 \geq 0$, $x_1-x_2+x_3 = 1$.

The objective function can be written

$$f(\lambda,x) = 1/2\left[(x_1+\lambda)^2+x_2^2+(x_3-2\lambda)^2\right] - 5/2\ \lambda^2,$$

and thus for any λ the solution x will be that point of the constraint set closest to the point $(-\lambda, 0, 2\lambda)$. This is illustrated in Figure 1 for $\lambda = 1$, and in Figure 2 for general $\lambda \geq 0$.

For this problem

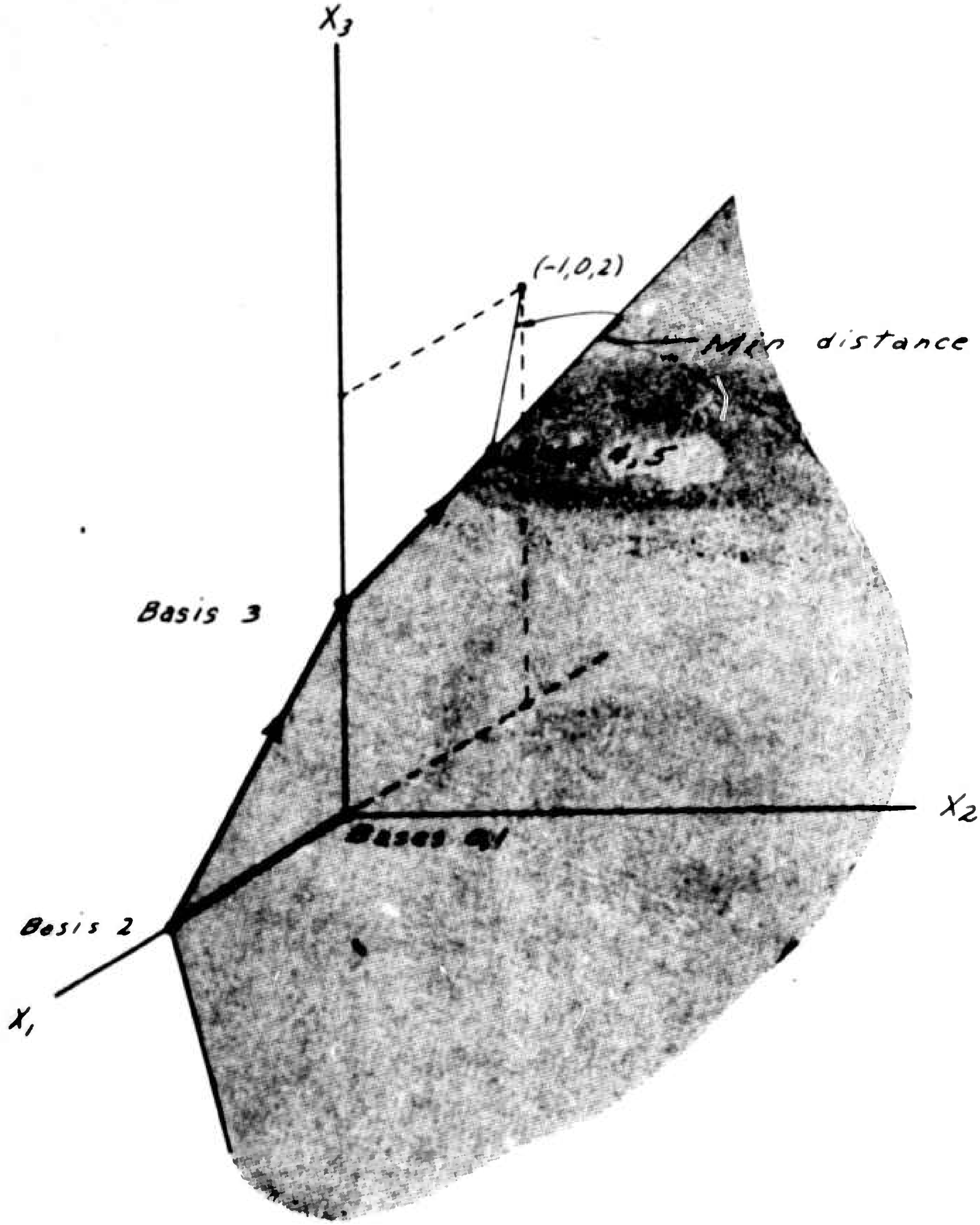

FIGURE 1: SHORT FORM

$$A = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \end{bmatrix} \qquad p = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$$

Since C is positive definite, the short form will solve the problem for any λ. Taking $\lambda = 1$, formulas (10,11) give the initial array for the short form.

x_1	x_2	x_3	v_1	v_2	v_3	u	z_1^1	z_2^1	z_3^1	z_1^2	z_2^2	z_3^2	w	
1	-1	1	0	0	0	0	0	0	0	0	0	0	1	= 1
1	0	0	-1	0	0	1	1	0	0	-1	0	0	0	= -1
0	1	0	0	-1	0	-1	0	1	0	0	-1	0	0	= 0
0	0	1	0	0	-1	1	0	0	1	0	0	-1	0	= 2

Although there is considerable degeneracy in the problem, the minimization of $\sum z_k$ proceeds without hitch. Below are given the values of the variables in the successive steps (only the values of the basic variables are given). The variable u is introduced first, since it will be in every basis. Since it is unrestricted, it might have been eliminated from the system, but we have left it in.

Basis	x_1	x_2	x_3	v_1	v_2	v_3	u	z_1^1	z_2^1	z_3^1	z_1^2	z_2^2	z_3^2	w	$\sum z$	
0									0	2	1			1		Initia
1							2		2		3			1		
2	1						2		2		4				6	
3			1				1		1		2				3	
4		$\frac{1}{2}$	$\frac{3}{2}$				$\frac{1}{2}$				$\frac{3}{2}$				$\frac{3}{2}$	
5		$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$			$\frac{1}{2}$								0	

The path traced out by x— from (1,0,0) to (0,0,1) to $(0,\frac{1}{2},\frac{3}{2})$ — is sketched in Figure 1.

For the long form, the initial array for the problem is

x_1	x_2	x_3	v_1	v_2	v_3	u	μ	z_1^1	z_2^1	z_3^1	z_1^2	z_2^2	z_3^2	w		
1	-1	1	0	0	0	0	0	0	0	0	0	0	0	1	=	1
1	0	0	-1	0	0	1	1	1	0	0	-1	0	0	0	=	0
0	1	0	0	-1	0	-1	0	0	1	0	0	-1	0	0	=	0
0	0	1	0	0	-1	1	-2	0	0	1	0	0	-1	0	=	0

The sequences of values are:

Basis	x_1	x_2	x_3	v_1	v_2	v_3	u	μ	z_1^1	z_2^1	z_3^1	z_1^2	z_2^2	z_3^2	w	
0									0	0	0				1	short form
1							0			0	0				1	
2	0						0				0				1	
3	0		0				0								1	
4	$\frac{1}{2}$		$\frac{1}{2}$				$-\frac{1}{2}$						$\frac{1}{2}$			
5	$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$		$-\frac{1}{2}$									
6			1		$\frac{1}{3}$		$-\frac{1}{3}$	$\frac{1}{3}$								
7			1	$\frac{1}{2}$			0	$\frac{1}{2}$								
8		t	$1+t$	$\frac{1}{2}+2t$			t	$\frac{1}{2}+t$								

$x^{\infty} = (0,1,1)$

The x-parts of these solutions are traced out in Figure 2.

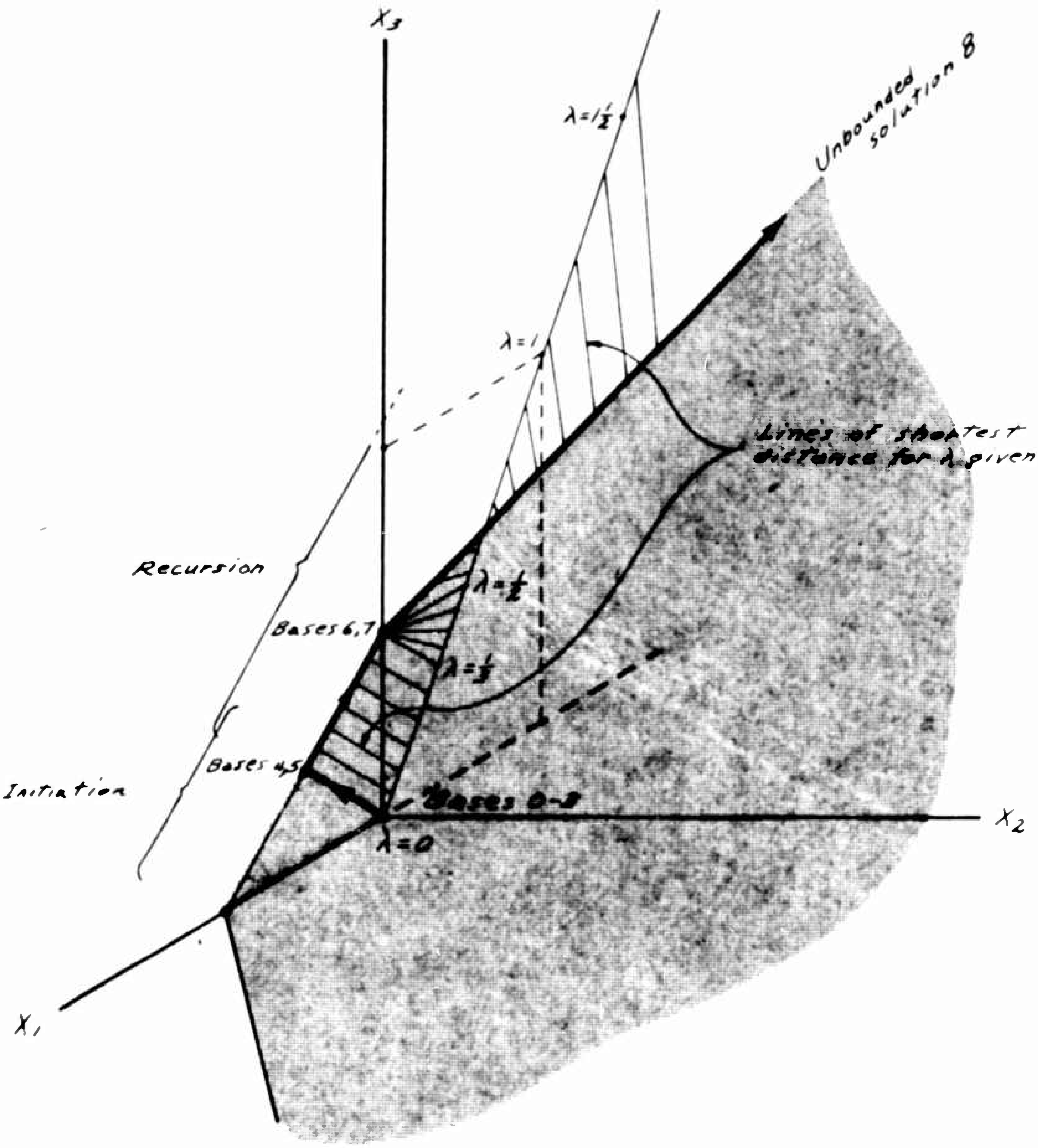

FIGURE 2: LONG FORM

For example, the solution of the problem for $\lambda = 1/4$ is given by (21) as an interpolation between the Basis 5 and Basis 6 solutions as follows:

$$x = \frac{1/4-1/3}{0-1/3}\ x^2 + \frac{0-1/4}{0-1/3}\ x^3 = 1/4x^2 + 3/4x^3 = (1/8,0,7/8).$$

The solution here for $\lambda = 1$ is given directly from Basis 8 (cf. formula 22) for $t = 1/2$ as $(0,1/2,3/2)$, the same answer the short form gave.

5. PROOFS

The burden of this Section is to prove the statements regarding the two terminations of the recursions of Section 3.

The initiation of the process ($v=0$) and the side condition (18) on the choice of x's and v's entering the basis have been designed to ensure that at each stage of both recursions we have $v'x=0$. It remains to see what occurs when, in either recursion, it is not possible to continue the indicated minimization under the side conditions. The theorem below gives what is needed for analysis of these conditions.

THEOREM 3

Let A, b, C be as in Section 1; let the matrix Q be n by n', q be 1 by n', and g be n by 1. Let $x\geq 0$, $v\geq 0$ such that $v'x=0$ be given. Denote by x_x those components of x which are positive, and by v_x the corresponding components of v (note $v_x=0$); denote by v_v the positive components of v, and by x_v the corresponding components of x (note $x_v=0$).

If the linear form

(23) qw

is minimal under the linear constraints

$$(24)\qquad \begin{aligned} v_x &= 0 \\ x_v &= 0 \end{aligned} \qquad \text{and}$$

$$(25)\qquad \begin{aligned} &Ax && = b \\ &Cx - Iv + A'u + Qw && = g, \end{aligned}$$

coefficients appear below the array. The columns corresponding to $x_v=0$, $v_x=0$ can be disregarded, since we insist that these variables vanish.

At the left of the array stand the variables of the linear programming problem <u>dual</u> to ours [8]. The coefficients of the linear form for the dual problem are those on the right of the array; the constraints for the dual problem, read vertically from the matrix, have their constant coefficients on the bottom of the array. The dual variables are unrestricted in sign since they are connected with equation constraints. The existence of these variables, satisfying the relations indicated below the matrix, is the consequence of the duality theorem for linear programming [8], as is the equality of the two objective functions. (Note particularly that where a variable turns out to be non-zero, as in x_x, or is never restricted, as in u, the corresponding dual relation is an equality.)

In detail, these relations are:

$$
\begin{array}{lll}
\text{(a)} & sA_x + r_x\, C_{xx} + r_\delta\, C_{x\delta} + r_v\, C_{xv} & = 0 \\
\text{(b)} & sA_\delta + r_x\, C'_{x\delta} + r_\delta C_{\delta\delta} + r_v\, C_{\delta v} & \leq 0 \\
\text{(c)} & \quad - r_\delta\, I_\delta & \leq 0 \\
\text{(d)} & \quad - r_v\, I_v & = 0 \\
\text{(e)} & \qquad r_x\, A'_x + r_\delta\, A'_\delta + r_v\, A'_v & = 0 \\
\text{(f)} & \qquad [r_x r_\delta r_v]\ Q & \leq q
\end{array}
\tag{27}
$$

and

(g) $qw = sb + rg,$

which expresses the equality of the objectives.

then there exists r such that

$$rC = 0,\ Ar' = 0, \text{ and } qw = rg.$$

PROOF: (Note that (24) is precisely the linear expression of the basis restriction side condition (18).) The proof depends upon the detailed structure of the quantities x, v, u, w which yield the minimum of qw. We have already distinguished in the vectors x, v the corresponding parts $x_x > 0$, $v_x = 0$, and the corresponding parts $x_v = 0$, $v_v > 0$. There remain the corresponding parts x_δ, v_δ, which, although not positive, are not <u>required</u> to vanish by the constraints. In (26), below, the matrix for the constraints (25) is partitioned in accordance with this partitioning of x and v, first vertically, then horizontally in the naturally corresponding manner (such that -I partitions into diagonal matrices).

$$(26)\qquad \begin{array}{c|c|c|c|c|c|c|c|c|c}
 & x_x>0 & x_\delta \geq 0 & x_v = 0 & v_x = 0 & v_\delta \geq 0 & v_v > 0 & u & w \geq 0 & \\
\hline
s & A_x & A_\delta & A_v & 0 & 0 & 0 & 0 & 0 & = b \\
\hline
r_x & C_{xx} & C'_{x\delta} & C'_{xv} & -I_x & 0 & 0 & A'_x & & = g_x \\
r_\delta & C_{x\delta} & C_{\delta\delta} & C'_{\delta v} & 0 & -I_\delta & 0 & A'_\delta & Q & = g_\delta \\
r_v & C_{xv} & C_{\delta v} & C_{vv} & 0 & 0 & -I_v & A'_v & & = g_v \\
\hline
 & = & \geq & & & \geq & = & = & \geq & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q &
\end{array}$$

According to the hypothesis of the theorem, the values of the variables above this array minimize the linear form whose

Relations (d) and (c) yield at once

$$r_v = 0, \quad r_\delta \geq 0.$$

Dropping therefore r_v, multiplying (a) on the right by r_x' and (b) by r_δ', we have

$$sA_x r_x' + r_x C_{xx} r_x' + r_\delta C_{x\delta}' r_x' = 0,$$

$$sA_\delta r_\delta' + r_x C_{x\delta} r_\delta' + r_\delta C_{\delta\delta} r_\delta' \leq 0,$$

which are added to form

$$s\left[A_x r_x' + A_\delta r_\delta'\right] + \left[r_x r_\delta\right] \begin{bmatrix} C_{xx} & C_{x\delta} \\ C_{x\delta}' & C_{\delta\delta} \end{bmatrix} \left[r_x r_\delta\right]' \leq 0.$$

By (e), however, the first term here vanishes; and the matrix of C's in the second term, being a principal submatrix of a positive semidefinite matrix is itself positive semidefinite, so that the second term is in fact zero; whence, by lemma 1,

$$\begin{bmatrix} C_{xx} & C_{x\delta} \\ C_{x\delta}' & C_{\delta\delta} \end{bmatrix} (r_x r_\delta)' = 0, \text{ or}$$

$$(28) \quad \begin{aligned} C_{xx} r_x' + C_{x\delta}\, r_\delta' &= 0, \\ C_{x\delta}' r_x' + C_{\delta\delta} r_\delta' &= 0. \end{aligned}$$

Equation (a) then yields just

$$sA_x = 0,$$

whence we have

$$sb = s\left[A_x x_x + A_\delta x_\delta + A_v x_v\right] = sA_x x_x = 0,$$

which, by (g), proves the theorem, letting $r = \left[r_x \; r_\delta \; r_v \right]$ and noting (27e) and (28).

This theorem can be applied to the short form computation by letting

(29) $Q = E$, $q = (1,..,1)$, and $g = -\lambda p'$.

When, in the course of minimizing $\sum Z_k$, it is not possible to reduce it under the basis restrictions, the hypothesis of Theorem 3 will be satisfied, and we will have

$$\sum Z_k = qw = rg = -\lambda rp',$$

with $rC = 0$. Thus in either the case that C is positive definite — when necessarily $r = 0$ — or in the case that $\lambda = 0$ we have $\sum Z_k = 0$, so that the hypothesis of Theorem 2 is satisfied, and the terminal x solves the quadratic problem.

Having reduced Z to zero, we maintain this in the long form and proceed to minimize $-\lambda$. Theorem 3 is applied here by letting

(30) $Q = p'$, $q = -1$, $g = 0$.

If the long form recursion ends in a finite minimum for $-\lambda$, the hypothesis of Theorem 3 is satisfied, whence we conclude

$$-\lambda = qw = rg = 0;$$

$-\lambda$ has in fact not been reduced. Two cases are thus possible: (i) no step lowering $-\lambda$ can be taken; and (ii) $-\lambda$ can be reduced to $-\infty$.

Case i: Here we must make use of the achievable non-degeneracy of the constraints (26) for this system, which asserts that m+n of the variables in any solution are positive. Since $\lambda=0$, m of these are in u, and the remaining n in x_x and v_v; x_δ, v_δ are empty. Since $Ar'=0$, $Cr'=0$, and, from (27f) $pr'\leq -1$, we have that for any t

$$\text{(31)}\quad A(x+tr') = b,$$
$$C(x+tr') - v + A'u = 0.$$

It follows from non-degeneracy that $r\geq 0$; for otherwise we should have, for some $t\geq 0$, $x+tr'$ satisfying (25) but vanishing in one more component than does x. Thus $x+tr'$ is feasible for all $t\geq 0$, and

$$f(\lambda, x+tr') = \lambda px + \tfrac{1}{2}x'Cx + \lambda pr't.$$

Since $pr'\leq -1$, $f(\lambda, x+tr') \to -\infty$ as $t\to\infty$ for any $\lambda>0$, and the desired minimum is $-\infty$.

Case ii: The values of λ are not bounded. Since only a finite number of bases is available, a sequence of basic solutions (x^i, v^i, u^i, μ^i), $i=1,..,g$ will be produced, and finally $(x^{g+1}, v^{g+1}, u^{g+1})$ such that $(x^g+tx^{g+1}, v^g+tv^{g+1}, u^g+tu^{g+1}, \mu^g t)$ is a solution for all $t\geq 0$. Owing to the basis restriction (18), we will have these relations:

$$\text{(32)}\quad v^i x^i = v^i x^{i+1} = v^{i+1} x^i = v^{i+1} x^{i+1},$$

$$\mu^i < \mu^{i+1} \quad (i=1,..,g).$$

Given now $\mu^i \leq \lambda \leq \mu^{i+1}$, the point

$$x = \frac{\mu^{i+1} - \lambda}{\mu^{i+1} - \mu^i} \; x^i + \frac{\lambda - \mu^i}{\mu^{i+1} - \mu^i} \; x^{i+1},$$

being a convex combination of x^i and x^{i+1}, is feasible; and it is easy to check that, letting v and u be respectively the same combinations of v^i, v^{i+1} and u^i, u^{i+1}, the resulting triple satisfies Theorem 3, so that x yields the desired minimum. If on the other hand $\lambda \geq \mu^g$, the triple $x^g + (\lambda - \mu^g)x^{g+1}$, $v^g + (\lambda - \mu^g)v^{g+1}$, $u^g + (\lambda - \mu^g)\, u^{g+1}$ satisfies Theorem 3, so that $x^g + (\lambda - \mu^g)\, x^{g+1}$ is the answer.

6. COMPUTATION

Although we have presented the procedure above only for the case of constraints of the form $Ax = b$, $x \geq 0$, relying on the fact that all types of linear inequality constraints may be written in this form, in practical computations there are several devices which will serve to reduce the magnitude of the problem when other types of constraints are given. We shall give these below without proofs of their effectiveness; such proofs follow closely along the line of those of Section 4.

Let the constraints of the stated problem be

$$\begin{aligned}
(33)\qquad A_{11}\,x_1 + A_{12}\,x_2 \phantom{{}+y_2} &= b_1 \\
A_{21}\,x_1 + A_{22}\,x_2 + y_2 &= b_2 \\
A_{31}\,x_1 + A_{32}\,x_2 - y_3 &= b_3 \\
x_1,\ y_2,\ y_3 &\geq 0.
\end{aligned}$$

(The second and third lines of (33) are the usual formulations for the constraints $\leq$ and $\geq$.) The new system of linear constraints (corresponding to (9) of Section 2) will be

$$(34)\qquad \begin{array}{|cc|cc|c|cccc|c}
x_1\geq 0 & x_2 & y_2\geq 0 & y_3\geq 0 & v_1\geq 0 & u_1 & u_2\geq 0 & u_3\geq 0 & \lambda & \\ \hline
A_{11} & A_{12} & & & & & & & & = b_1 \\
A_{21} & A_{22} & I & & & & & & & = b_2 \\
A_{31} & A_{32} & & -I & & & & & & = b_3 \\ \hline
 & & & & -I & A'_{11} & A'_{21} & A'_{31} & p'_1 & = 0 \\
 & C & & & & A'_{12} & A'_{22} & A'_{32} & p'_2 & = 0 \\ \hline
\end{array}$$

The algorithm proceeds as before, with rule (18) of Section 3 strengthened to:

(35) If $(x_1)_k$ is in the basis, do not admit $(v_1)_k$, and vice versa; if $(y_2)_k$ is in the basis, do not admit $(u_2)_k$, and vice versa; if $(y_3)_k$ is in the basis, do not admit $(u_3)_k$, and vice versa.

In this formulation, it is seen that the number of equations in the stated problem is augmented only by the number of non-slack variables in the problem. A further reduction is evidently possible: since the variables x_2 and u_1 are not restricted, they could be algebraically eliminated from the system, along with an equal number of equations in which they have nonzero coefficients (this reduction could also have been performed with the u of (9)). The eliminations would leave a number of equations equal to the total number of components of x_1, y_2, and y_3 for this generalized problem, and just the number, n, of components of x in the simpler problem — in any case, the number of inequality constraints in the original problem. (This might seem odd if, for example, there were no inequality constraints in the stated problem; but then the operations of eliminating the unrestricted variables — <u>all</u> the variables — would be precisely those of solving for x and u in the classical Lagrange Multiplier solution.)

While the elimination is simple to perform, we have not employed it in calculation, for the reason that it is not

likely, in problems whose matrices have many zero entries, to decrease very greatly the number of nonzero entries; and it is this latter number which, to a considerable extent, determines speed of computation in sophisticated versions of the simplex method. This must be kept in mind when estimating the relative efficiencies of procedures of this sort; in this procedure, the bulk of the data consists of the entries of C and twice those of A. In the large problem described below, these data were only 1660 in number, although the resulting linear programming problem had 204 equations and 714 variables.

A revision of the SHARE Linear Programming code for the IBM 704 computer has been made for the solution of quadratic programming problems. This code can be used for either the short form or the long form as described above, or in an alternate version which first gets the solution to the linear problem obtained by dropping the quadratic form, and then proceeds to obtain the solutions for all $\lambda \geq 0$. The code has been used on a variety of problems, the largest of which, concerned with the allocation of a strategic material, had 90 constraints and 192 variables, 78 of which were "slacks." This problem required 359 simplex method changes-of-basis during 230 minutes for the complete long form solution.

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